Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 4

Exercise 1 *(4 Points)*
The following exercise is about a version of Borel-Cantelli's lemma: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ with

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \mathbb{P}\left(A_n^c \cap A_{n+1}\right) < \infty.$$

Prove that $\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0$.

Exercise 2 *(4 Points)*

(a) Show that a family of sets $(A_i)_{i\in I}$ is independent if and only if the family of random variables $(\mathbb{1}_{\{A_i\}})_{i\in I}$ is.

(b) Let X and Y be independent and exponentially distributed random variables with parameters $\theta > 0$ and $\lambda > 0$, respectively. Prove that

$$\mathbb{P}(X < Y) = \frac{\theta}{\theta + \lambda}.$$

Exercise 3 *(4 Points)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n}$ and $Y = \sum_{m=1}^{M} b_m \mathbb{1}_{B_m}$ with $a_n, b_n \geq 0$ and $A_n, B_m \in \mathcal{F}$ such that

$$a_n \neq a_k, \quad b_n \neq b_k, \quad A_n \cap A_k = \emptyset, \quad B_n \cap B_k = \emptyset$$

for $n \neq k$. Prove that the following are equivalent

- X, Y are independent.
- $\mathbb{P}(A_n \cap B_m) = \mathbb{P}(A_n)\mathbb{P}(B_m)$ holds for all $1 \leq n \leq N$ and all $1 \leq m \leq M$.

Exercise 4 *(4 Points, talk)*
Prepare a talk on the construction and uniqueness problem for the conditional expecation.